Categorical actions in geometry and representation theory

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Representation theory of $\mathfrak{sl}_2$

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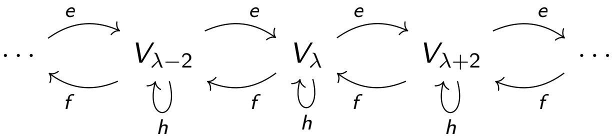

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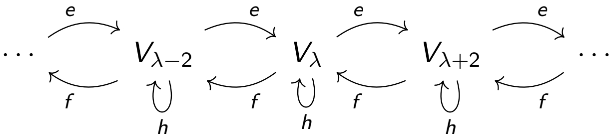

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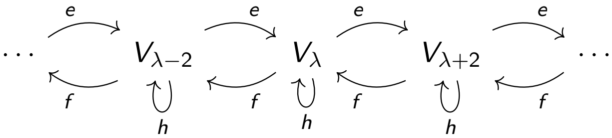

Conversely, this can be used to define a representation as:

- A collection of vector spaces V_λ, $\lambda \in \mathbb{Z}$,
- with linear maps $e \colon V_\lambda \to V_{\lambda+2}$, $f \colon V_\lambda \to V_{\lambda-2}$ and $h \colon V_\lambda \to V_\lambda$,
- such that $h|_{V_\lambda} = \lambda$,
- and $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

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- Idea: Take a problem that uses linear algebra and replace all vector spaces by (triangulated) categories.
- Why? Well-chosen categories can provide additional structure.
- There are procedures to decategorify, i.e. to get back the vector spaces.
- Choosing good categories is an art. Often geometry is useful.

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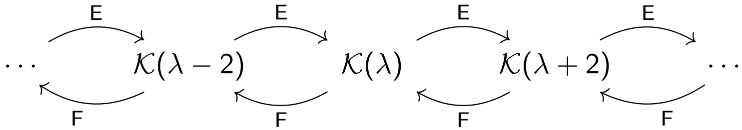

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$$\mathsf{EF}|_{\mathcal{K}(\lambda)} - \mathsf{FE}|_{\mathcal{K}(\lambda)} = \lambda$$

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$$\mathsf{EF}|_{\mathcal{K}(\lambda)} - \mathsf{FE}|_{\mathcal{K}(\lambda)} = \bigoplus_{\lambda} \mathsf{Id}_{\mathcal{K}(\lambda)}$$

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$$\mathsf{EF}|_{\mathcal{K}(\lambda)} \cong \mathsf{FE}|_{\mathcal{K}(\lambda)} \oplus \bigoplus_{\lambda} \mathsf{Id}_{\mathcal{K}(\lambda)}$$

General categorical actions

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- Categories $\mathcal{K}(\lambda)$ are indexed by a "weight lattice" (depending on $\mathfrak{g}$).
- More pairs of functors E_i, F_i between these categories.
- More relations ("Serre relations") between the functors.

A new use of categorical actions

- Start with some categories associated to spaces of importance to representation theory.
- Complete them to a categorical $\mathfrak{sl}_n$-action.
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Theorem (Cautis–K.)

Any categorical (quantum, affine) $\mathfrak{sl}_n$-action has an "abelian refinement": Abelian categories $\mathcal{A}(\lambda) \subseteq \mathcal{K}(\lambda)$ such that E_i, F_i restrict to

$$\mathcal{A}(\lambda) \underset{\mathsf{F}_i}{\overset{\mathsf{E}_i}{\rightleftarrows}} \mathcal{A}(\lambda + \alpha_i)$$

Note that the $\mathcal{K}(\lambda)$ are typically derived categories of sheaves and the E_i, F_i are not exact.

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- We produce many new interesting abelian categories, connected by the restrictions of the E_i, F_i. (Technically, we produce t-structures.)
- These are expected to have very nice properties.
- Some special cases of these categories were already known (with different, often complicated, constructions) and have been used to prove deep theorems in representation theory.

My main interests

- t-structures on categories of sheaves
  - perverse sheaves
  - perverse-coherent sheaves
  - exotic sheaves
- support theories of Benson–Iyengar–Krause and Arinkin–Gaitsgory
  - support theory for D-modules on stacks?
  - Hochschild cohomology of dg-categories
  - Geometric Langlands
- D-modules in log geometry
- geometry inspired by representation theory

Example: $\mathfrak{g} = \mathfrak{sl}_n$

$$\Lambda \cong \mathbb{Z}^{n-1} = \{(k_1, \dots, k_n) : \sum k_i = n\}, \qquad \alpha_i = (0, \dots, -1, 1, \dots, 0)$$

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Define spaces

$$X(\underline{k}) = \left\{ (V_\bullet, A) : 0 = V_0 \overset{k_1}{\subseteq} V_1 \overset{k_2}{\subseteq} \cdots \overset{k_n}{\subseteq} V_n = \mathbb{C}^n,\ AV_i \subseteq V_{i-1} \right\}$$

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The functors E_i and F_i are defined via correspondences along

$$X(\underline{k},\ \underline{k} + \alpha_i) = \{((V_\bullet, A), (V'_\bullet, A)) \in X(\underline{k}) \times X(\underline{k} + \alpha_i) : \\ V_j = V'_j \text{ for } j \neq i,\ V'_i \subseteq V_i\}.$$