D-MODULES: EXERCISE SHEET 3

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For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ we let $G_i\mathscr{M}$ denote the largest submodule G of $\mathscr{M}$ with $\dim \mathrm{Ch}(G) \leq \dim X + i$. This defines an increasing filtration G_* on $\mathscr{M}$, called the *Gabber filtration*.

We remark that such a largest submodule $G_i\mathscr{M}$ indeed exists because $\mathscr{D}_X$ is noetherian and for $\mathscr{N}_1, \mathscr{N}_2 \subseteq \mathscr{M}$ we have $\mathrm{Ch}(\mathscr{N}_1 \cup \mathscr{N}_2) = \mathrm{Ch}(\mathscr{N}_1) \cup \mathrm{Ch}(\mathscr{N}_2)$. It is however not obvious that this filtration behaves functorially. The following series of exercises shows that the filtration G is indeed useful.

Exercise 1. Suppose $\mathscr{M} \in \mathbf{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ is a coherent $\mathscr{D}_X$-module. Then $\mathbb{D}_X(H^\ell(\mathbb{D}_X\mathscr{M})) \in \mathrm{D}^{\geq \ell}_{\mathrm{coh}}(\mathscr{D}_X)$ for all $\ell \in \mathbb{Z}$.

Denote by $\tau_{\geq i}$ the truncation functors in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Note that for any $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ there is a canonical map $\mathscr{M} \to \tau_{\geq i}\mathscr{M}$.

Exercise 2. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ let $S_i\mathscr{M}$ be the image of the canonical morphism

$$H^0(\mathbb{D}_X\tau_{\geq -i}\mathbb{D}_X\mathscr{M}) \to H^0(\mathbb{D}_X \circ \mathbb{D}_X\mathscr{M}) = \mathscr{M}.$$

Show that his map is injective and S is an increasing filtration on $\mathscr{M}$. It is called the *Sato–Kashiwara filtration*. [Hint: consider the distinguished triangle $H^{-(i+1)}(\mathbb{D}_X\mathscr{M})[i+1] \to \tau_{\geq -(i+1)}(\mathbb{D}_X\mathscr{M}) \to \tau_{\geq -i}(\mathbb{D}_X\mathscr{M})$ and apply duality.]

Note that the Sato–Kashiwara filtration is functorial: Any morphism $\mathscr{M} \to \mathscr{N}$ induces a morphism $S_i\mathscr{M} \to S_i\mathscr{N}$.

Exercise 3. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Show that the Gabber and Sato–Kashiwara filtrations of $\mathscr{M}$ agree, i.e.,

$$G_i\mathscr{M} = S_i\mathscr{M} \quad \text{for all } i \in \mathbb{Z}.$$

In particular, we can functorially associate to each coherent $\mathscr{M}$ its maximal holonomic submodule.

Exercise 4. Let $\mathscr{M} \in \mathbf{Mod}_{\mathrm{qc}}(\mathscr{D}_X)$ and let $U \subset X$ be open. Assume we are given a holonomic submodule $\mathscr{N}$ of $\mathscr{M}$ of $\mathscr{M}|_U$. Show that there exists a holonomic submodule $\widetilde{\mathscr{N}}$ of $\mathscr{M}$ such that $\widetilde{\mathscr{N}}|_U = \mathscr{N}$.

Exercise 5. Without using Gabber's theorem, show Bernstein's inequality. [Hint: Use the above filtrations to reduce to showing $\dim \mathrm{Ch}(\mathscr{M}) \geq \dim X$ for all non-zero $\mathscr{M} \in \mathbf{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$. Then induct on the dimension of the support of $\mathscr{M}$.]