D-MODULES: EXERCISE SHEET 3

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For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \) we let \( G_i \mathcal{M} \) denote the largest submodule \( G \) of \( \mathcal{M} \) with \( \dim \text{Ch}(G) \leq \dim X + i \). This defines an increasing filtration \( G_* \) on \( \mathcal{M} \), called the Gabber filtration.

We remark that such a largest submodule \( G_i \mathcal{M} \) indeed exists because \( \mathcal{D}_X \) is noetherian and for \( \mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M} \) we have \( \text{Ch}(\mathcal{N}_1 \cup \mathcal{N}_2) = \text{Ch}(\mathcal{N}_1) \cup \text{Ch}(\mathcal{N}_2) \). It is however not obvious that this filtration behaves functorially. The following series of exercises shows that the filtration \( G \) is indeed useful.

Exercise 1. Suppose \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X) \) is a coherent \( \mathcal{D}_X \)-module. Then \( \mathcal{D}_X(H^i(\mathcal{D}_X \mathcal{M})) \in \mathcal{D}_{\geq i}^{\text{coh}}(\mathcal{D}_X) \) for all \( i \in \mathbb{Z} \).

Denote by \( \tau_{\geq i} \) the truncation functors in \( \mathcal{D}^{\text{coh}}(\mathcal{D}_X) \). Note that for any \( \mathcal{M} \in \mathcal{D}^{\text{coh}}(\mathcal{D}_X) \) there is a canonical map \( \mathcal{M} \to \tau_{\geq i} \mathcal{M} \).

Exercise 2. For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \) let \( S_i \mathcal{M} \) be the image of the canonical morphism

\[
H^0(\mathcal{D}_X \tau_{\geq -i} \mathcal{D}_X \mathcal{M}) \to H^0(\mathcal{D}_X \mathcal{D}_X \mathcal{M}) = \mathcal{M}.
\]

Show that his map is injective and \( S \) is an increasing filtration on \( \mathcal{M} \). It is called the Sato–Kashiwara filtration. [Hint: consider the distinguished triangle \( H^{-i+1}(\mathcal{D}_X \mathcal{M})[i + 1] \to \tau_{\geq -i+1} \mathcal{D}_X \mathcal{M} \to \tau_{\geq -i} \mathcal{D}_X \mathcal{M} \) and apply duality.]

Note that the Sato–Kashiwara filtration is functorial: Any morphism \( \mathcal{M} \to \mathcal{N} \) induces a morphism \( S_i \mathcal{M} \to S_i \mathcal{N} \).

Exercise 3. Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module. Show that the Gabber and Sato–Kashiwara filtrations of \( \mathcal{M} \) agree, i.e.,

\[
G_i \mathcal{M} = S_i \mathcal{M} \quad \text{for all } i \in \mathbb{Z}.
\]

In particular, we can functorially associate to each coherent \( \mathcal{M} \) its maximal holonomic submodule.

Exercise 4. Let \( \mathcal{M} \in \text{Mod}_{\text{qc}}(\mathcal{D}_X) \) and let \( U \subset X \) be open. Assume we are given a holonomic submodule \( \mathcal{N} \) of \( \mathcal{M} \) of \( \mathcal{M}|_U \). Show that there exists a holonomic submodule \( \tilde{\mathcal{N}} \) of \( \mathcal{M} \) such that \( \tilde{\mathcal{N}}|_U = \mathcal{N} \).

Exercise 5. Without using Gabber’s theorem, show Bernstein’s inequality. [Hint: Use the above filtrations to reduce to showing \( \dim \text{Ch}(\mathcal{M}) \geq \dim X \) for all non-zero \( \mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X) \). Then induct on the dimension of the support of \( \mathcal{M} \).]