D-MODULES: EXERCISE SHEET 3

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For a filtered k-algebra A with an increasing filtration $F_\bullet$ we define its *Rees ring* $R(A)$ as the submodule $\bigoplus_n (F_nA)z^n$ of $A \otimes_k k[z]$. Similar, for a filtered module M we define its *Rees module* $R(M)$. (Recall our assumptions on filtrations from the lectures.)

Exercise 1. Let A be a filtered k-algebra.

(i) Show that the Rees constructions is a functor from the category of filtered A-modules to the category of graded $R(A)$-modules.

(ii) Show that for any filtered A-module M, one has canonical isomorphisms $R(M)/zR(M) \cong \operatorname{gr} M$ and $R(M)/(z-1)R(M) \cong M$.

(iii) Show that any graded $R(M)$ without z torsion is obtained from a filtered A-module via the above functor.

In the same way we can apply the Rees construction to filtered sheaves of rings, and hence in particular to D-modules.

For a morphism $f\colon X \to Y$ smooth complex varieties define a functor

$$f_\bullet^R\colon \mathrm{D}^+(R(\mathscr{D}_X)) \to \mathrm{D}^+(R(\mathscr{D}_Y)), \qquad f_\bullet^R \mathscr{N} = \mathbb{R}f_*(R(\mathscr{D}_{Y\leftarrow X}) \overset{\mathbb{L}}{\otimes}_{R(\mathscr{D}_X)} \mathscr{N}).$$

Exercise 2. Let $f\colon X \to Y$ be a morphism of smooth complex varieties. Show the following:

(i) If f is proper, then $f_\bullet^R$ preserves coherence.

(ii) For any $R(\mathscr{D}_X)$-module $\mathscr{N}$ and any integer i there exists a canonical equivalence

$$H^i(f_\bullet^R \mathscr{N}) \Big/ (z-1)H^i(f_\bullet^R \mathscr{N}) \cong H^i(f_\bullet(\mathscr{N}/(z-1)\mathscr{N})).$$

(iii) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then the quotient of $H^i(f_\bullet^R R(M))$ by its z-torsion is the Rees module associated to some good filtration on $H^i(f_\bullet M)$.

Recall that for a morphism $f\colon X \to Y$ we have an associated diagram of cotangent bundles

$$\begin{array}{ccc} & f^*(T^*Y) & \\ {}^{\rho}\swarrow & & \searrow^{\varpi} \\ T^*X & & T^*Y. \end{array}$$

Exercise 3. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and $f\colon X \to Y$ a proper morphism of smooth complex varieties. The goal of this exercise is to show Kashiwara's estimate:

$$\operatorname{Ch}(f_\bullet \mathscr{M}) \subseteq \varpi(\rho^{-1} \operatorname{Ch} \mathscr{M}).$$

(i) Define a functor $f_\bullet^{\mathrm{gr}}\colon \mathbf{Mod}(\operatorname{gr} \mathscr{D}_X) \to \mathbf{Mod}(\operatorname{gr} \mathscr{D}_Y)$ as $\mathbb{R}\varpi_* \circ \mathbb{L}\rho^*$. Show that $\operatorname{supp} H^i(f_\bullet^{\mathrm{gr}} \operatorname{gr} \mathscr{M}) \subseteq \varpi(\rho^{-1} \operatorname{Ch} \mathscr{M})$.

(ii) Show that the z-torsion submodule of $H^i(f_\bullet^R(R(\mathscr{M})))$ is given by $\ker z^\ell$ for some fixed ℓ.

(iii) Set $\operatorname{gr}_{[\ell]} \mathscr{M} = \bigoplus_k F_k\mathscr{M}/F_{k-\ell}\mathscr{M}$. Define a functor $f_\bullet$ for $\operatorname{gr}_{[\ell]} \mathscr{D}_X$-modules analogous to the one for $\mathscr{D}_X$-modules. Show that $\operatorname{gr}_{[\ell]} H^i(f_\bullet M)$ is a $\operatorname{gr}_{[\ell]} \mathscr{D}_Y$-subquotient of $H^i(f_\bullet \operatorname{gr}_{[\ell]} \mathscr{M})$.

(iv) Filter $\operatorname{gr}_{[\ell]} \mathscr{D}_X$ by the finite filtration

$$G_j \operatorname{gr}_{[\ell]} \mathscr{D}_X = \bigoplus_k F_{k+j-\ell}\mathscr{D}_X \Big/ F_{k-\ell}\mathscr{D}_X\,, \qquad 0 \le j \le \ell,$$

and similarly for $\operatorname{gr}_{[\ell]} \mathscr{D}_X$-modules. Show that $\operatorname{gr}^G \operatorname{gr}^F_{[\ell]} \mathscr{D}_X \cong \operatorname{gr}^F \mathscr{D}_X[u]/u^\ell$ with u in G-degree 1. Use the spectral sequence associated to the filtration G to show that the support of $\operatorname{gr}^G H^j(f_\bullet \operatorname{gr}^F_{[\ell]} \mathscr{M})$ is contained in $\operatorname{supp} H^i(f_\bullet^{\mathrm{gr}} \mathscr{M})$.

(v) Show that the support of $\operatorname{gr}^G \operatorname{gr}^F_{[\ell]} H^i(f_\bullet \mathscr{M})$ is contained in the support of $\operatorname{gr}^G H^j(f_\bullet \operatorname{gr}^F_{[\ell]} \mathscr{M})$. Use this to conclude the argument.